PHYSICAL REVIEW E VOLUME 58, NUMBER 5 NOVEMBER 1998

Elementary derivation of Spitzer's asymptotic law for Brownian windings and some of its physical applications

Arkady L. Kholodenko*
375 H. L. Hunter Laboratories, Clemson University, Clemson, South Carolina 29634-1905

(Received 6 March 1998)

A simple derivation of Spitzer's asymptotic law for Brownian windings [Trans. Am. Math. Soc. **87**, 187 (1958)] is presented along with its generalizations. These include the cases of planar Brownian walks interacting with a single puncture and Brownian walks on a single truncated cone with variable conical angle interacting with the truncated conical tip. Such situations are typical in the theories of quantum Hall effect and $2+1$ quantum gravity, respectively. They also have some applications in polymer physics. Extension of these results to the multiple punctured case is also briefly discussed. It is technically associated with some results known in the context of string and conformal field theories and theories of quantum chaos.
[S1063-651X(98)50911-7]

PACS number(s): 02.50.−r, 05.40.+j, 61.41.+e

In 1958, based on earlier work by Levy [1], Spitzer [2] had obtained the asymptotic probability distribution $P(x)$ for the winding angle θ for the planar Brownian motion. If $z(t)=x(t)+iy(t)$ is a two-dimensional Wiener process, then it is of interest to study distributions of $|z(t)|$ and $\arg z(t)=\theta(t)$. For large times t, Spitzer obtained his famous Cauchy-type distribution for $\theta(t)$ given by

$$P\left(x=\frac{2\theta}{\ln t}\right)dx=\frac{1}{\pi}\frac{1}{1+x^2}\,dx. \tag{1}$$

Equation (1) is obtained under the assumption that the random walker begins his travel at some point z_0 in the z plane other than the origin, i.e., $x=y=0$. Then, the angle θ is measured with respect to the line that joins z_0 with the origin. This problem is of interest in polymer physics [3–5], since it represents the benchmark problem for the study of entanglements. In mathematical literature the same phenomenon is described in terms of recurrence and transience. For example, it is well known [6] that one- and two-dimensional Brownian motions are recurrent (that is, the random walk visits time and again its starting point), while three-dimensional motion is transient (that is, there is a nonzero probability that the walker will not return to the origin). For the random walk on a once-punctured z plane, Lyons and McKean [7] had demonstrated that the walk is recurrent while for the twice-punctured plane the walk is transient. In the language of polymers this means that the polymer lying in the z plane will not be entangled with another polymer, placed perpendicular to this plane, while it will become entangled if there are at least two polymers that intersect the z plane at two distinct points. The planarity of the above problem is actually not too essential, as was explained in Ref. [5]. In the case of quantum mechanics the once-punctured plane problem is directly associated with the Aharonov-Bohm (AB) effect [8]. The AB effect in the presence of two punctures was studied in Ref. [9]. The methods of Ref. [9] cannot be generalized to the case of more than two punctures and do not provide any information about the recurrence and/or transience. At the same time, the methods used by McKean and Lyons [7] can be used for the case of more than two punctures but are not widely known in physics literature. They had been recently mentioned in Ref. [5] in connection with some topological problems arising in polymer physics. In physics literature random walks on a multiply punctured plane were extensively studied in connection with problems related to the quantum Hall effect (QHE) and anyonic superconductivity [8,10], while in mathematics literature the same problem was recently extensively studied by Pitman and Yor [11,12]. To our knowledge, no attempt had been made to establish connections between these two formalisms (see also Ref. [7]). In this Rapid Communication we would like to make the first step towards this comprehensive goal.

Let us begin with the well-known expression for the distribution function for the planar random walk given by

$$G(\mathbf{r}_1,\mathbf{r}_2;t)=\frac{1}{2\pi t}\exp\left(-\frac{(\mathbf{r}_1-\mathbf{r}_2)^2}{2t}\right), \tag{2}$$

where $\mathbf{r}_1(\mathbf{r}_2)=\{x_1(x_2),y_1(y_2)\}$. With respect to the origin in the z plane the polar system of coordinates can be used. In this system of coordinates Eq. (2) can be rewritten as

$$G(r_1,r_2,\Delta\theta;t)=\frac{1}{2\pi t}\exp\left\{-\frac{r_1^2+r_2^2}{2t}\right\}\sum_{m=-\infty}^{\infty}e^{im\Delta\theta}I_m(z), \tag{3}$$

where $\Delta\theta=\theta_1-\theta_2$, $z=2r_1r_2/t$ and $I_m(z)=I_{-m}(z)$ is the modified Bessel function. The above distribution function can be used for study of either the radial or the angular distributions or both. Suppose, we are interested in the angular distribution function only [in view of Eq. (1)]. Then, using Eq. (3), it is convenient to introduce the normalized distribution function defined according to the following prescription:

*Electronic address: string@mail.clemson.edu

$$f(z,\Delta\theta)=\frac{G(r_1,r_2,\Delta\theta;t)}{G(r_1,r_2,0;t)}=\frac{1}{I_0(z)}\sum_{m=-\infty}^{\infty}e^{im\Delta\theta}I_{|m|}(z). \tag{4}$$

The Fourier transform of such defined distribution function can now be obtained in a standard way as

$$f(z,\alpha)=\int_{-\infty}^{\infty}d\Delta\theta e^{-i\alpha\Delta\theta}f(z,\Delta\theta)=\frac{I_{|\alpha|}(z)}{I_0(z)}. \tag{5}$$

Let us now choose $r_2=\hat{r}\sqrt{t}+r_1$. This choice is motivated by known scaling properties of Brownian motion [13]. Then, for large t, one obtains $z\simeq 2r_1\hat{r}/\sqrt{t}$. For fixed $\hat{r}$ and r_1 and $t\rightarrow\infty$ one surely expects $z\rightarrow 0$. This observation allows us to use a known asymptotic expansion for $I_{|\alpha|}(z)$ for small z's with the result for $f(z,\alpha)$ (valid for small z's or large t's):

$$f(z,\alpha)\approx\exp\left(-\frac{|\alpha|}{2}\ln t\right). \tag{6}$$

The inverse Fourier transform of Eq. (6) leads us to the result given by Eq. (1), i.e., $f(z,\Delta\theta)=P(x)$, where $x=2\Delta\theta/\ln t$.

Thus the obtained result can be easily generalized now. For example, instead of considering random walks on the flat once-punctured plane we can consider the same problem on the surface of a cone. This type of problem is of interest in connection with the study of $2+1$ quantum gravity [14,15]. It is also known [15,16] that the above conical problem is equivalent to the planar random walk problem in the wedge (the conical angle is simply related to that of the wedge). In the most general case our walk may be allowed to interact with the edges of the wedge [2] or, in the case of the wedge angle equal to 2π, with the puncture located at the origin. The analog of the distribution function, Eq. (3), is known to be [16,17]

$$G(r_1,r_2,\Delta\theta;t)=\frac{1}{\beta t}\exp\left(-\frac{r_1^2+r_2^2}{2t}\right)\times\sum_{m=-\infty}^{\infty}e^{i2\pi(m+\delta)\Delta\theta/\beta}I_{(2\pi/\beta)|m+\delta|}(z). \tag{7}$$

For $\beta=2\pi$ and $\delta=0$ Eq. (7) reduces to Eq. (3) as required. The wedge angle β lies between 0 and 2π while the statistics-changing parameter δ is responsible for the polymer puncture interactions, as is explained in Refs. [5,18], or for the interaction with the flux tube if the magnetic language is being used [8].

By analogy with Eq. (4), we obtain

$$f_\beta^\delta(z,\Delta\theta)=\frac{1}{I_{(2\pi/\beta)|\delta|}(z)}\sum_{m=-\infty}^{\infty}e^{i2\pi m(\Delta\theta/\beta)}I_{(2\pi/\beta)|m+\delta|}(z). \tag{8}$$

Upon Fourier transforming this expression we obtain

$$f_\beta^\delta(z,\alpha)=\frac{I_{|\alpha+(2\pi/\beta)\delta|}(z)}{I_{(2\pi/\beta)|\delta|}(z)}. \tag{9}$$

Repeating the same chain of arguments that had led us to Eq. (6), we obtain now

$$f_\beta^\delta(z,\alpha)\simeq\exp\left[-\frac{1}{2}\left(\left|\alpha+\frac{2\pi}{\beta}\delta\right|-\frac{2\pi}{\beta}|\delta|\right)\ln t\right]. \tag{10}$$

To perform the inverse Fourier transform of Eq. (10) is nontrivial. Indeed, we have

$$f_\beta^\delta(\Delta\theta,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\alpha\,\exp\left\{i\alpha\Delta\theta-\frac{1}{2}\left[\sqrt{\alpha^2+\left(\frac{2\pi}{\beta}\delta\right)^2}-\frac{2\pi}{\beta}|\delta|\right]\ln t\right\}. \tag{11}$$

The integrals of this type are known in the context of quantum field theory [19] and had also been used recently in polymer physics problems [20]. By introducing new variable $\alpha=|2\pi\delta/\beta|\sinh\varphi$ into Eq. (11) it is transformed into

$$f_\beta^\delta(\Delta\theta,t)=\frac{\delta}{\beta}\exp\left\{\frac{\pi}{\beta}|\delta|\ln t\right\}\int_{-\infty}^{\infty}d\varphi\,\cosh\varphi\times\exp\left(-\frac{1}{2}(\cosh\varphi)\frac{2\pi}{\beta}|\delta|\ln t+i\Delta\theta\frac{2\pi}{\beta}|\delta|\sinh\varphi\right). \tag{12}$$

The exponent inside the integral in Eq. (12) can be transformed as follows:

$$-\frac{1}{2}(\cosh\varphi)\frac{2\pi}{\beta}|\delta|\ln t+i\Delta\theta\frac{2\pi}{\beta}|\delta|\sinh\varphi=-\sqrt{a^2+\omega^2}\cosh(\varphi+\varphi_0),$$

where $a=(\pi/\beta)|\delta|\ln t$ and $\omega=\Delta\theta(2\pi/\beta)|\delta|$ so that $\cosh\varphi_0=a/\sqrt{a^2+\omega^2}$ and $\sinh\varphi_0=-i\omega/\sqrt{a^2+\omega^2}$. The use of these results in Eq. (12) allows us to rewrite it in the equivalent form,

$$f_\beta^\delta(\Delta\theta,t)=\frac{\delta(2\pi/\beta)|\delta|\ln t}{\beta\sqrt{a^2+\omega^2}}\exp\left(\frac{\pi}{\beta}|\delta|\ln t\right)\int_0^{\infty}d\varphi\,\cosh\varphi\times\exp(-\sqrt{a^2+\omega^2}\cosh\varphi)\equiv\Phi_\beta^\delta(t)K_1(\sqrt{a^2+\omega^2}), \tag{13}$$

where $K_1(x)$ is the modified Bessel function with known asymptotic expansions: $K_1(x)\simeq 1/x$ for $x\rightarrow 0$, and $K_1(x)\simeq\sqrt{\pi/2x}e^{-x}$ for $x\rightarrow\infty$. Using these expansions, the following asymptotic results for the distribution function $f_\beta^\delta(\Delta\theta,t)$ are obtained.

(i) $\delta\rightarrow 0$ and β is fixed and nonzero. In this case we recover Spitzer's law, Eq. (1), as required.

(ii) $\delta\rightarrow\infty$ but $\Delta\theta$ is finite. Then we obtain ($x=2\Delta\theta/\ln t$),

$$f_\beta^\delta(x)dx\simeq\frac{1}{2}\sqrt{\frac{\delta}{2\beta}\ln t}\,\frac{dx}{(1+x^2)^{3/4}}\exp\left(-\left|\frac{\beta\ln t}{8\pi\delta}\right|x^2\right). \tag{14}$$

The result obtained is in complete accord (up to numerical prefactors) with that obtained in Ref. [21] using methods of conformal field theory and, more recently, in Ref. [22], by direct numerical simulations.

(iii) δ and β are finite but $\Delta\theta\rightarrow\infty$ and $\Delta\theta\gg\ln t$. Then we obtain

$$f_{\beta}^{\delta}(x)dx\simeq\frac{1}{2}\sqrt{\frac{\delta}{2\beta}\ln t}\,\exp\left(\left|\frac{\pi\delta}{\beta}\right|\ln t\right)\times\exp\left(-|x|\left|\frac{\pi\delta}{\beta}\right|\ln t\right)\frac{dx}{|x|^{3/2}}. \tag{15}$$

This result is in formal agreement with that obtained by Rudnick and Hu [4] and Desbois [23]. In Ref. [4] only the leading exponential factor was obtained, while the result, Eq. (15), differs from that obtained in Ref. [23] by an extra time-dependent factor. No connections with conical singularities or QHE problems were made in either Ref. [4] or Ref. [23].

With the results just obtained, we are ready now to make connections with works of Ito and McKean [24] and Lyons and McKean [7] (see also Ref. [25]). We begin by stating the result proved by Levy [1] and refined by others [13]. Let $z(t)$ be some planar Brownian motion which started, say, at $z=0$ (at $t=0$). Then, the motion associated with z is obtained with the help of some analytic function $f(z)$ is also Brownian. This is equivalent to saying that the planar Brownian motion is conformally invariant. Let us illustrate this fact in the example of once-punctured plane $\mathbf{R}^2-\mathbf{0}$.

The diffusion equation (in dimensionless units) on $\mathbf{R}^2-\mathbf{0}$ can be written in a usual form as

$$\frac{\partial f}{\partial t}=\frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f, \tag{16}$$

where the factor 1/4 is chosen for convenience. It is useful to rewrite Eq. (16) in terms of complex variables. Simple calculation produces

$$\frac{\partial f}{\partial t}=\frac{\partial^2 f}{\partial z\partial\bar{z}}. \tag{17}$$

Since this equation is not defined for $z=\bar{z}=0$, we introduce new complex variable w through $z=\exp w$. Unlike z, our new variable $w=u+iv$ is defined for the entire complex w plane. In terms of the w variable Eq. (17) can be rewritten as

$$\frac{\partial f}{\partial t}=\frac{1}{2}e^{-2u}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)f. \tag{18}$$

The equation obtained describes Brownian motion on a simply connected covering space $\tilde{M}$, which is a Riemann surface for the logarithmic function. The result just obtained coincides with that discussed in the book by Ito and McKean; e.g. see Ref. [24]. Ito and McKean argue (without explicit demonstration) that, based on the results of Levy, the diffusion equation (18) can be converted into standard form given by Eq. (16) by replacing Brownian time t by another (actually random) time T which is properly chosen. Let us demonstrate how this can actually be done. To this purpose, let us consider the Langevin-type equation written in the form of Ito,

$$dx(s)=a(x(s))ds+\sigma(x(s))dw(s). \tag{19}$$

The corresponding backward Kolmogorov-Fokker-Planck equation can be written now as

$$\frac{\partial}{\partial s_0}P(x,s|x_0,s_0)=-a\frac{\partial}{\partial x_0}P+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x_0^2}P. \tag{20}$$

Introduce ''new'' time $T(s)$ according to equation

$$s=\int_0^{T(s)}dtg(x(t)). \tag{21}$$

Therefore, $ds=g(x(T))dT$. Substitution of this result into Eq. (19) and use of Ito stochastic calculus [13] produces

$$dx(T)=a(x(T))g(x(T))dT+\sqrt{g(x(T))}\sigma(x(T))dw(T). \tag{22}$$

A new diffusion coefficient can be selected now [in view of Eqs. (19),(20)] as

$$D_{\text{new}}=\frac{1}{2}g\sigma^2. \tag{23}$$

Looking at Eq. (18),(21) and selecting

$$s=\int_0^{T(s)}dt\;e^{2u(t)} \tag{24}$$

produces, in view of Eq. (23), $D_{\text{new}}=1/2$. Equation (24) is in complete agreement with the result of Ito and McKean [24], where it was presented without derivation. Surely, $T(s)$ is random time so that solution of Eq. (16) will depend upon random time. This is somewhat inconvenient. To correct this inconvenience it is useful to think about calculable observables. In view of Eq. (5), we are interested in the Fourier-transformed angular distribution function $f(z,\alpha)$. It can be shown [24] that this task is equivalent to finding averages of the type

$$f(z,\alpha)=\langle e^{i\alpha\Delta\theta}\rangle=\left\langle\exp\left(-\frac{\alpha^2}{2}\int_0^T d\tau g[x(\tau)]\right)\right\rangle, \tag{25}$$

where $\langle\,\rangle$ denotes the averaging with the help of a Gaussian-like propagator for the free ''particle.'' Equation (25) is a special case of the famous Feynman-Kac formula [26]. The transition from first to second average in Eq. (25) is associated with the Cameron-Martin-Girsanov-type formula for changes of variables inside of path integrals [13,27]. Equation (25) is associated with the Schrödinger-like differential equation. In view of Eqs. (21) and (24), it can be easily demonstrated, using standard methods of quantum mechanics [28], that such an obtained Schrödinger-like equation is just that for the modified Bessel functions, e.g., $I_m(x)$, etc. Its solution, indeed, leads us to Eq. (5).

Based on these results, immediate generalizations are possible. For instance if we were looking for the distribution function $f(z,\alpha)$ for a twice-punctured plane, we would need the uniformizing function which will help us to write the diffusion equation on the simply connected universal covering surface $\widetilde{M}$. This surface is known to be the Riemann surface for the punctured torus [25]. Classical motion on such a surface is chaotic and quantum description of such motion was considered in Ref. [29], where no attempts were made to relate it to Spitzer's results. Already for two punctures study of transience and recurrence is highly nontrivial [7,25]. For two or more punctures the task of finding the uniformizing function can be reduced to that of finding all solutions of the corresponding Fuchsian-type equations [30], which are identical in the form to that known in string and conformal field theories for the correlation functions [31]. It remains a challenging problem to extend Spitzer-like results to multiply connected surfaces, to connect these results with those of Pitman and Yor [11,12] and with those known in the theories of QHE [8,10].

Note added in proof. Recently, two additional references came to our attention. In Ref. [32] the reader can find more up-to-date (as compared to Pitman and Yor's) references on Brownian windings, while Ref. [33] containing detailed applications of the above ideas to QHE.

[1] P. Levy, *Processus Stochastiques et Movement Brownien* (Gauthier-Villars, Paris, 1948).
[2] F. Spitzer, Am. Math. Soc. Trans. **87**, 187 (1958).
[3] S. Edwards, Proc. Phys. Soc. London **91**, 513 (1967).
[4] J. Rudnick and Y. Hu, Phys. Rev. Lett. **60**, 712 (1988); J. Phys. A **20**, 4421 (1987).
[5] A. Kholodenko and Th. Vilgis, Phys. Rep. **298**(5,6), 251 (1998).
[6] F. Spitzer, Principles of Random Walk (Van Nostrand, Princeton, NJ, 1964).
[7] T. Lyons and H. McKean, Adv. Math. **51**, 212 (1984).
[8] A. Shapere and F. Wilczek, Geometric Phases in Physics (World Scientific, Singapore, 1990).
[9] S.-Wu Qian, Zhi-Y. Gu, and Guo-Q. Xie, J. Phys. A **30**, 1273 (1997).
[10] F. Wilczek, Fractional Statistics and Anyon Superconductivity (World Scientific, Singapore, 1990).
[11] J. Pitman and M. Yor, Ann. Prob. **14**, 733 (1986).
[12] J. Pitman and M. Yor, Ann. Prob. **17**, 965 (1989).
[13] R. Durrett, Brownian Motion and Martingales (Wadsworth Inc., Belmont, CA, 1984).
[14] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1994).
[15] A. Kholodenko (unpublished).
[16] J. Dowker, J. Phys. A **10**, 115 (1977).
[17] S. Deser and R. Jackiw, Commun. Math. Phys. **118**, 495 (1988).
[18] A. Kholodenko, J. Douglas, and D. Bearden, Phys. Rev. E **49**, 2206 (1994).
[19] N. Bogoliubov and D. Shirkov, Introduction to the Theory of Quantized Fields (Wiley, New York, 1980).
[20] A. Kholodenko and Th. Vilgis, Phys. Rev. E **50**, 1257 (1994).
[21] B. Duplantier and H. Saleur, Phys. Rev. Lett. **60**, 2343 (1988).
[22] T. Prellberg and D. B. Drossel, Phys. Rev. E **57**, 2045 (1998).
[23] J. Desbois, J. Phys. A **23**, 3099 (1990).
[24] K. Ito and H. McKean, *Diffusion Processes and Their Sample Paths* (Springer-Verlag, Berlin, 1965).
[25] H. McKean and D. Sullivan, Adv. Math. **51**, 203 (1984).
[26] M. Freidlin and A. Wentzell, Random Perturbations of Dynamical Systems (Springer-Verlag, Berlin, 1984).
[27] I. Gelfand and A. Yaglom, J. Math. Phys. **1**, 48 (1961).
[28] R. Feynman and A. Hibbs, Quantum Mechanics and Path Integrals (McGraw Hill, New York, 1965).
[29] T. Shigehara, Phys. Rev. E **50**, 4357 (1994).
[30] J. Hempel, Bull. London Math. Soc. **20**, 97 (1988).
[31] P. Ginsparg and G. Moore, in Recent Directions in Particle Theory (World Scientific, Singapore, 1993), pp. 277–469.
[32] Z. Shi, Ann. Prob. **26**, 112 (1998).
[33] A. Carey, K. Hannebuss, V. Mathai, and P. McCann, Commun. Math. Phys. **190**, 629 (1998).